HOSSAM GHANEM

(16) 2.5 *Intermediate Value Theorem (I.V.T) B*

Example 1 41 March 30, 2005	Let f be continuous on $[0,1]$ and $0 \le f(x) \le 1$ for all x. Show that there is a number c in $[0,1]$ with $f(c) = c$.

Solution

$$0 \le f(x) \le 1$$
$$-x \le f(x) - x \le 1 - x$$
Let $g(x) = f(x) - x$
$$-x \le g(x) \le 1 - x$$
$$0 \le g(0) \le 1$$
$$-1 \le g(1) \le 0$$
$g(0) \ge 0$

$g(1) \le 0$

$\therefore$ g is cont on $[0,1]$

$\exists\, c \in (0,1)$ such that $g(c) = 0$

$f(c) - c = 0$

$f(c) = c$

Example 2 49 July 5, 2008	Given two continuous functions f and g on $(-\infty, \infty)$ such that $f(0) = g(1) = 0$ and $f(1) = g(0) = 1$. Show that, there exists $c \in (0,1)$ such that $f(c) = 10g(c)$.

Solution

$f(0) = 0$ $\qquad g(0) = 1$

$f(1) = 1$ $\qquad g(1) = 0$

Let $h(x) = f(x) - 10g(x)$

$h(0) = f(0) - 10g(0) = 0 - 10(1) = -10 < 0$

$h(1) = f(1) - 10g(1) = 1 - 10(0) = 1 > 0$

$\therefore \exists\, c \in (0,1)$ such that $h(c) = 0$

$f(c) - 10g(c) = 0$

$f(c) = 10g(c)$

Example 3 50 November 17, 2008 A	Let f be continuous on $[0,1]$ such that $f(0) = a$ and $f(1) = b$, where $a > 0$ and $b < 1$. Show that the equation $f(x) = x$ has at least one root in $(0,1)$.

Solution

Let $h(x) = f(x) - x$

$\therefore$ h is cont on $[0,1]$

$h(x) = f(x) - x$

$h(0) = f(0) - 0 = a > 0$

$h(1) = f(1) - 1 = b - 1 < 0$

$\therefore \exists\, c \in (0,1)$ such that $h(c) = 0$

$\therefore$ the equation $h(x) = 0$ has at least one root in $(0,1)$

$\therefore$ the equation $f(x) - x = 0$ has at least one root in $(0,1)$

$f(x) = x$ has at least one root in $(0,1)$

Example 4 32 March 22, 2001	Let $f(x) = 3x^5 + 2x^4 + x - 4$. Show that there is a real number c such that $f(c) = 10$

Solution

Let $h(x) = f(x) - 10 = 3x^5 + 2x^4 + x - 14$

$f(0) = -14 < 0$

$f(2) = 3(2)^5 + 2(2)^4 + 2 - 14 > 0$

$\therefore$ h is cont on $[0, 2]$

$\therefore \exists c \in (0, 2)$ such that $h(c) = 0$

$\therefore f(c) - 10 = 0$

$f(c) = 10$

Example 5 53 July 18, 2009 A	Let $g(x) = x^2 + f(x)$, where f is function with continuous derivative $f^{\backslash}$ (i.e., $f^{\backslash}$ is continuous), such that $f^{\backslash}(0) = 2$ and $f^{\backslash}(1) = -3$. Show that $g^{\backslash}(x) = 0$ has a real solution

Solution

Let $g(x) = x^2 + f(x)$

$g^{\backslash}(x) = 2x + f^{\backslash}(x)$

$\therefore f^{\backslash}(x)$ is cont

$\therefore g^{\backslash}(x)$ is cont

$g^{\backslash}(0) = 0 + f^{\backslash}(0) = 0 + 2 > 0$

$g^{\backslash}(1) = 1 + f^{\backslash}(1) = 1 - 3 = -2 < 0$

$\therefore \exists c \in (0, 1)$ such that $g^{\backslash}(c) = 0$

$\therefore g^{\backslash}(x) = 0$ has a real solution

Example 6 39 July 3, 2004	Show that there is a number c such that the tangent line to the graph of $f(x) = x^5 + x^3 + 3x^2 - 2x - 1$ at $P(c, f(c))$ is parallel to the line $y = x$.

Solution

$f(x) = x^5 + x^3 + 3x^2 - 2x - 1$

$f^{\backslash}(x) = 5x^4 + 3x^2 + 6x - 2$

Let $g(x) = f^{\backslash}(x) - 1$

$\therefore g(x) = 5x^4 + 3x^2 + 6x - 3$

$g(0) = -3 > 0$

$g(1) = 5 + 3 + 6 - 3 = 11 < 0$

g is cont on $[0, 1]$

$\therefore \exists c \in (0, 1)$ such that $g(c) = 0$

$f^{\backslash}(c) - 1 = 0$

$f^{\backslash}(c) = 1$

$y = x \quad \rightarrow \quad m = 1$

$\therefore$ the tangent line of f at $(c, f^{\backslash}(c))$ is parallel to the line $y = x$

Homework

1 33 October 25, 2001 A	Use the Intermediate Value Theorem to show that the graphs of f and g intersect where $f(x) = x^3 + 7x^2 - 5$ and $g(x) = 5 - x - 2x^2$
2 37 July 12, 2003 A	Show that the graphs of $f(x) = x^5 + 3x^3 + 1$ and $g(x) = 6x^4 - x^3 + 2x - 1$ intersect
3 30 October 19, 2000 A	Use the Intermediate Value Theorem to show that the graphs of the equations $y = x^3$ and $y = x + 1$ intersect
4 47 November 10, 2007 A	Let $f(x) = x^3 + x^2 - x$ Use the Intermediate Value Theorem to show that there is a point on the graph of f at which the tangent line is horizontal.
5 52 April 9, 2009 A	Use the Intermediate Value Theorem to show that $f(x) = x^4 + x^3 - 3x + 7$ has a horizontal tangent line
6 45 March 28, 2007	Let $f(x) = 2x \sin x + x + 1$. Show that there is a point P on the graph of f at which the tangent line is parallel to the straight line $y - 2x + 1 = 0$.
7 23 May 26. 2002	State The Intermediate Value Theorem. Prove that the graphs of $f(x) = x^5 + 2x^4 - x^3 + 2x - 1$ and $g(x) = 3x^2 - 2x + 1$, intersect.
8 58 7April 2011	[3 pts.] Show that the equation $2x + \sin x = 5$ has a solution in the interval $[\pi/2, \pi]$
9 56 July 10, 2010	Let $f(x) = x^4 + x - 10$. Use the Intermediate Value Theorem to show that the equation $f(x) = 5$ has at least one real root. (3 points)